University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO PROBLEM SET 9
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Reading:
105 Notes 10.1-10.3, 11.1-11.4
Hand & Finch 8.4-8.12

1.
Three equal point masses are located at $(a, 0, 0)$, $(0, 2a, 2a)$, and $(0, 2a, a)$. About the origin, find the principal moments of inertia and a set of principal axes.
Solution:
For a number of point masses, the expression for the inertia tensor becomes

$$I_{ij} = \sum m_k \left(|\mathbf{x_k}|^2 \delta_{ij} - x_{ki} x_{kj} \right)$$

So:

$$\begin{aligned}
I_{xx} &= m \left(0^2 + 0^2 + (2a)^2 + (2a)^2 + (2a)^2 + a^2 \right) \\
&= 13ma^2 \\
I_{yy} &= m \left(a^2 + 0^2 + 0^2 + (2a)^2 + 0^2 + a^2 \right) \\
&= 6ma^2 \\
I_{zz} &= m \left(a^2 + 0^2 + 0^2 + (2a)^2 + 0^2 + (2a)^2 \right) \\
&= 9ma^2 \\
I_{yx} = I_{xy} &= -m \left((a)(0) + (0)(2a) + (0)(2a) \right) \\
&= 0 \\
I_{zx} = I_{xz} &= -m \left((a)(0) + (0)(2a) + (0)(a) \right) \\
&= 0 \\
I_{zy} = I_{yz} &= -m \left((0)(0) + (2a)(2a) + (2a)(a) \right) \\
&= -6ma^2
\end{aligned}$$

So the full inertia tensor is

$$\mathcal{I} = ma^2 \begin{pmatrix} 13 & 0 & 0 \\ 0 & 6 & -6 \\ 0 & -6 & 9 \end{pmatrix}$$

The principal moments and the principal axes are the eigenvalues and eigenvectors of this matrix. Skipping the explicit calculation, the principal moments are $13ma^2$ and $\frac{3}{2}(5 \pm \sqrt{17})ma^2$, and the (unnormalized) eigenvectors are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \frac{1}{4}(1 \pm \sqrt{17}) \\ 1 \end{pmatrix}.$$

2.
Consider a rigid body that is plane, *i.e.* it lies in the plane $z = 0$.
(a)
Prove that the z axis is a principal axis.
Solution:
If the z axis is a principal axis, then $\hat{z}$ will be an eigenvector of the inertia tensor. Thus we need to show that $I_{xz} = I_{yz} = 0$. That way, $\mathcal{I}\hat{z} = I_{xz}\hat{x} + I_{yz}\hat{y} + I_{zz}\hat{z} = I_{zz}\hat{z}$. Suppose $\sigma(x, y)$ is the surface mass density, so that the volume mass density is $\rho(x, y, z) = \sigma(x, y)\delta(z)$. Then:

$$\begin{aligned}
I_{xz} &= -\int \rho(x, y, z)\, y\, z\, dx\, dy\, dz \\
&= -\int \sigma(x, y)\, dx\, dy \int z\, \delta(z)\, dz \\
&= 0
\end{aligned}$$

$I_{yz} = 0$ by a similar argument. Thus $\hat{z}$ is a principal axis.

(b)
Prove that the diagonalized inertia tensor for this plane rigid body has the largest element equal to the sum of the two smaller elements.
Solution:
Let's assume we've chosen a coordinate system

whose axes are the principal axes. Then the three moments of inertia are

$$I_{xx} = \int \sigma(x,y)\, y^2\, dx\, dy$$
$$I_{yy} = \int \sigma(x,y)\, x^2\, dx\, dy$$
$$I_{zz} = \int \sigma(x,y)\, (x^2+y^2)\, dx\, dy \; = I_{xx} + I_{yy}$$

3.
Design a solid right circular cylinder so that if it is rotated about *any* axis that passes through its center of mass, it will continue to rotate about that axis without wobbling.
Solution
We know from the Euler equations that if a body has all of its principal moments of inertia equal to one another, then the components of the angular velocity vector will be constant, ie the body will not wobble. Therefore, we must choose the height to width ratio to make these moments equal. Choose the origin to be at the center of mass of the cylinder, with the z axis along the cylinder's axis. Then, by symmetry, $I_{xx} = I_{yy}$, and all off diagonal elements vanish. Note that if the cylinder has mass m, height h, and radius R, then the density $\rho = \frac{m}{\pi R^2 h}$.

$$\begin{aligned} I_{zz} &= \int_{-h/2}^{h/2} dz \int_0^{2\pi} d\phi \int_0^R r\, dr\, \rho(x^2+y^2) \\ &= \frac{m}{\pi R^2 h}\int_{-h/2}^{h/2} dz \int_0^{2\pi} d\phi \int_0^R r^3\, dr \\ &= \frac{1}{2} m R^2 \\ I_{xx} &= \int_{-h/2}^{h/2} dz \int_0^{2\pi} d\phi \int_0^R r\, dr\, \rho(y^2+z^2) \end{aligned}$$
$$= \frac{m}{\pi R^2 h}\int_{-\frac{h}{2}}^{\frac{h}{2}} dz \int_0^{2\pi} d\phi \int_0^R r\, dr\, (r^2 \sin^2\phi + z^2)$$
$$= \frac{1}{2} m \left(\frac{1}{2}R^2 + \frac{1}{6}h^2\right)$$

We must have

$$\begin{aligned} I_{yy} = I_{xx} &= I_{zz} \\ R^2 &= \frac{1}{2}R^2 + \frac{1}{6}h^2 \\ R &= \frac{h}{\sqrt{3}} \end{aligned}$$

4.
Assume that the earth is a rigid solid sphere that is rotating about an axis through the North Pole. At $t = 0$ a mountain of mass 10^{-9} the earth's mass is added at north latitude 45°. The mountain is added "at speed" so that the earth's angular velocity ω is the same before and immediately after the mountain's addition.

Describe the subsequent motion of the rotation axis with respect to the North Pole. What is the velocity of its intersection with the earth's surface, in miles per year?
Solution:
Let's choose our axes to be the principal axes of the whole system (earth plus mountain.) Specifically, let $\hat{x}_3$ point in the direction of the mountain, and let $\hat{x}_1$ point east and $\hat{x}_2$ point north. Then the moments of inertia about these axes are

$$\begin{aligned} I_1 = I_2 &= \frac{2}{5} M R^2 + m R^2 \\ I_3 &= \frac{2}{5} M R^2 \end{aligned}$$

where M and R are the earth's mass and radius, and $m = 10^{-9} M$ is the mountain's mass. Let $\Delta I = I_1 - I_3$. Euler's equations are:

$$\begin{aligned} I_1 \dot{\omega}_1 &= \Delta I \omega_2 \omega_3 \\ I_1 \dot{\omega}_2 &= -\Delta I \omega_1 \omega_3 \\ I_3 \dot{\omega}_3 &= 0 \end{aligned}$$

ω_3 is constant, by the third equation. If we define $\Omega = \Delta I \omega_3 / I_1$, then the first two equations become

$$\begin{aligned} \dot{\omega}_1 &= \Omega \omega_2 \\ \dot{\omega}_2 &= -\Omega \omega_1 \; . \end{aligned}$$

Combine them to get $\ddot{\omega}_1 = -\Omega^2 \omega_1$. This is the equation for a harmonic oscillator, so $\omega_1(t)$ is a linear combination of $\sin \Omega t$ and $\cos \Omega t$. Since $\omega_1(0) = 0$, we must have $\omega_1(t) = A \sin \Omega t$

for some A. Then, taking the time derivative, we get $\omega_2(t) = A\cos\Omega t$. At t=0, we have $\omega_2 = \omega_3 = \omega/\sqrt{2}$, so $A = \omega/\sqrt{2}$.

$$\vec{\omega}(t) = \frac{\omega}{\sqrt{2}}\left(\hat{x}_1 \sin\Omega t + \hat{x}_2 \cos\Omega t + \hat{x}_3\right) .$$

Note that $|\vec{\omega}|$ is constant. The direction of $\vec{\omega}$ traces a circular path about the mountain, starting at the north pole and getting as far south as the equator. Let's figure out its speed along the earth's surface. Let $\hat{\omega}$ be a unit vector in the direction of $\vec{\omega}$: $\hat{\omega} = \vec{\omega}/\omega$. Then the point at which the angular velocity vector intersects the earth's surface is $\mathbf{q} = R\hat{\omega}$. In our chosen coordinate system,

$$\mathbf{q} = \frac{R}{\sqrt{2}}\left(\hat{x}_1 \sin\Omega t + \hat{x}_2 \cos\Omega t + \hat{x}_3\right)$$

The speed of this point is

$$\begin{aligned} |\dot{\mathbf{q}}| &= \left|\frac{R\Omega}{\sqrt{2}}(\hat{x}_1 \cos\Omega t - \hat{x}_2 \sin\Omega t)\right| \\ &= \frac{R\Omega}{\sqrt{2}} = \frac{\Delta I \omega R}{2I_1} \end{aligned}$$

$\Delta I = mR^2$, and $I_1 = \left(\frac{2}{5}M + m\right)R^2 \approx \frac{2}{5}MR^2$. The earth's angular frequency is $\omega = 7.29 \times 10^{-5} s^{-1}$ and its radius is $R = 6.38 \times 10^6$ m, so

$$\begin{aligned} |\dot{\mathbf{q}}| &= \frac{5}{4}\frac{m}{M}R\omega = 5.81 \times 10^{-7}\frac{\text{m}}{\text{s}} \\ &= 1.14 \times 10^{-2}\frac{\text{miles}}{\text{year}} \end{aligned}$$

5.

Assume that the earth is a rigid solid ellipsoid of revolution, rotating about its symmetry axis $\hat{\mathbf{x}}_3$, and that it has $1 - (I_2/I_3) = -0.0033$ (actually the earth *bulges* at the equator, so that this quantity is really positive). Two equal mountains are placed opposite each other on the equator "at speed", so that ω is the same immediately afterward. What fraction of the earth's mass must each mountain have in order to render the earth's rotation barely unstable with respect to small deviations of $\hat{\omega}$ from the $\hat{\mathbf{x}}_3$ axis?

Solution:

Use a coordinate system with $\hat{\mathbf{x}}_3$ pointing towards the north pole. Assume the mountains are added to the earth's surface along the positive and negative $\hat{\mathbf{x}}_1$ axis. Then the inertia tensor is diagonal both before and after the addition of the mountains. The earth's inertia tensor has diagonal elements (I_2, I_2, I_3), and the mountains contribute $(0, 2mR^2, 2mR^2)$, so the total inertia tensor has diagonal elements $(I_2, I_2 + 2mR^2, I_3 + 2mR^2)$ along the three coordinate directions. Before the mountains are added, the moment about the $\hat{\mathbf{x}}_3$ axis is the smallest. We know that in general a rotation about one of the principal axes is unstable if the moment of inertia about the axis is in between the other two. So instability sets in when

$$I_3 + 2mR^2 > I_2$$

(The other condition, $I_3 + 2mR^2 < I_2 + 2mR^2$ is always satisfied, since $I_3 < I_2$.) Rearrange this equation to get

$$m > \frac{I_2 - I_3}{2R^2} = 0.0033\frac{I_3}{2R^2} = 0.0033 \times \frac{1}{5}M$$

So the condition is $m > 0.00066M$.

6.

Consider an asymmetric body (principal moments $I_3 > I_2 > I_1$) initially rotating with $\vec{\omega}$ very close to the $\hat{x}_3$ axis.

(a)

Show that the projection of $\vec{\omega}(t)$ on the $\hat{x}_1 - \hat{x}_2$ plane describes an *ellipse*.

Solution:

Euler's equations are

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 \ . \end{aligned}$$

Differentiating the $\dot{\omega}_1$ equation with respect to time yields:

$$\begin{aligned} \ddot{\omega}_1 &= \frac{I_2 - I_3}{I_1}(\dot{\omega}_2\omega_3 + \dot{\omega}_3\omega_2) \\ &= \frac{I_2 - I_3}{I_1}\left(\omega_3\frac{I_3 - I_1}{I_2}\omega_3\omega_1 + \omega_2\frac{I_1 - I_2}{I_3}\omega_1\omega_2\right) \\ &= \frac{I_2 - I_3}{I_1}\omega_1\left({\omega_3}^2\frac{I_3 - I_1}{I_2} - {\omega_2}^2\frac{I_2 - I_1}{I_3}\right) \end{aligned}$$

Since $I_3 - I_1 > I_2 - I_1$, $\frac{1}{I_2} > \frac{1}{I_3}$, and ${\omega_3}^2 \gg {\omega_2}^2$, the first term in parentheses dominates the second. Therefore

$$\ddot{\omega}_1 \simeq -\left(\frac{I_3}{I_1} - 1\right)\left(\frac{I_3}{I_2} - 1\right){\omega_3}^2\omega_1$$

By a similar argument, we have:

$$\ddot{\omega}_2 \simeq -\left(\frac{I_3}{I_1} - 1\right)\left(\frac{I_3}{I_2} - 1\right){\omega_3}^2\omega_2$$

Thus both ω_1 and ω_2 execute simple harmonic motion with the same angular frequency

$$\Omega \equiv \left(\left(\frac{I_3}{I_1} - 1\right)\left(\frac{I_3}{I_2} - 1\right)\right)^{\frac{1}{2}}\omega_3 \ .$$

This will trace out what is known as a *Lissajous figure* in the ω_1-ω_2 plane, which is an ellipse if the frequencies of the two oscillators are identical, as they are in this case.

(**b**)

Calculate the ratio of the major and minor axes of the ellipse.

Solution:

Let

$$\begin{aligned}\omega_1 &= Re\left(\tilde{\omega}_1 e^{i\Omega t}\right)\\ \omega_2 &= Re\left(\tilde{\omega}_2 e^{i\Omega t}\right) \ .\end{aligned}$$

We choose to solve the complex equation of which this is the real part. Substituting this into the $\dot{\omega}_2$ Euler equation yields

$$\begin{aligned}I_2\left(i\Omega\tilde{\omega}_2 e^{i\Omega t}\right) &= \omega_3(I_3 - I_1)\left(\tilde{\omega}_1 e^{i\Omega t}\right)\\ \frac{\tilde{\omega}_2}{\tilde{\omega}_1} &= -\frac{i\omega_3}{\Omega}\frac{I_3 - I_1}{I_2}\end{aligned}$$

Thus $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are $\frac{\pi}{2}$ out of phase, and the ellipse axes are along ω_1 and ω_2. Thus the ratio of major and minor axes is:

$$\begin{aligned}\frac{|\tilde{\omega}_1|}{|\tilde{\omega}_2|} &= \frac{\omega_3}{\Omega}\frac{I_3 - I_1}{I_2}\\ &= \left(\frac{I_3 - I_1}{I_3 - I_2}\frac{I_1}{I_2}\right)^{\frac{1}{2}}\end{aligned}$$

This is < 1 or > 1 depending on the details of the inertia tensor.

7.

Consider a heavy symmetrical top with one point fixed. Show that the *magnitude* of the top's angular momentum about the fixed point can be expressed as a function only of the constants of motion and the polar angle θ of the top's axis.

Solution:

In terms of the Euler angles θ, ϕ, and ψ, the ω in the body axes can be written as

$$\begin{aligned}\omega = \ & \hat{\mathbf{x}}_\mathbf{1}\left(\dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi\right)\\ &+ \hat{\mathbf{x}}_\mathbf{2}\left(-\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi\right)\\ &+ \hat{\mathbf{x}}_\mathbf{3}\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\end{aligned}$$

and the inertia tensor as

$$\mathcal{I} = \begin{pmatrix} I & 0 & 0\\ 0 & I & 0\\ 0 & 0 & I_3\end{pmatrix} \ .$$

The angular momentum $\mathbf{L}$ is therefore

$$\begin{aligned}\mathbf{L} &= \mathcal{I}\omega\\ &= I\omega_1\hat{\mathbf{x}}_\mathbf{1} + I\omega_2\hat{\mathbf{x}}_\mathbf{2} + I_3\omega_3\hat{\mathbf{x}}_\mathbf{3}\\ L^2 &= I^2\left(\omega_1^2 + \omega_2^2\right) + I_3^2\omega_3^2\\ &= I^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + I_3^2\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2\end{aligned}$$

From the notes (eqs. 11.2 and 11.4) we have two conserved quantities, p_ψ and E:

$$\begin{aligned}p_\psi &= I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\\ E &= \frac{I}{2}\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{{p_\psi}^2}{2I_3} + mgh\cos\theta\end{aligned}$$

These can be rearranged to yield:

$$\begin{aligned}\dot{\phi}\cos\theta + \dot{\psi} &= \frac{p_\psi}{I_3}\\ \dot{\theta}^2 + \dot{\phi}^2\sin^2\theta &= \frac{2}{I}\left(E - \frac{{p_\psi}^2}{2I_3} - mgh\cos\theta\right)\end{aligned}$$

Inserting these quantities into the above expression for L^2, and simplifying, gives us:

$$L^2 = 2I\left(E - \frac{{p_\psi}^2}{2I_3} - mgh\cos\theta\right) + {p_\psi}^2$$

which is a function only of θ and constants of motion.

8.

Investigate the motion of the heavy symmetrical top with one point fixed for the case in which the axis of rotation is vertical (along $\hat{\mathbf{x}}_3$). Show that the motion is either stable or unstable depending on whether the quantity $4I_2Mhg/I_3^2\omega_3^2$ is less than or greater than unity. (If the top is set spinning in the stable configuration ("sleeping"), it will become unstable as friction gradually reduces the value of ω_3. This is a familiar childhood observation.)

Solution:

Eq. 11.4 in the notes tells us:

$$E = \frac{\left(p_\phi - p_\psi \cos\theta\right)^2}{2I\sin^2\theta} + \frac{I}{2}\dot{\theta}^2 + \frac{p_\psi^2}{2I_3} + Mgh\cos\theta$$

Using initial conditions $\theta = \phi = \psi = 0$, $\dot{\theta} = \dot{\phi} = 0$, and $\dot{\psi} = \omega_3$, along with equation 11.2 from the notes, gives us

$$\begin{aligned} p_\psi &= I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right) = I_3\omega_3 \\ p_\phi &= I\dot{\phi}\sin^2\theta + I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\cos\theta = I_3\omega_3 \end{aligned}$$

Inserting these into the above expression for E allows us to write $E = \frac{I}{2}\dot{\theta}^2 + V(\theta)$, where $V(\theta)$ is an effective potential and is defined as

$$V(\theta) = \frac{I_3^2\omega_3^2}{2I}\left(\frac{1-\cos\theta}{\sin\theta}\right)^2 + Mgh\cos\theta + \frac{1}{2}I_3\omega_3^2 \; .$$

If we take derivatives of $V(\theta)$ with respect to θ, we find

$$\begin{aligned} \frac{dV}{d\theta} &= \frac{I_3^2\omega_3^2}{I}\frac{\sin\theta}{\left(1+\cos\theta\right)^2} - Mgh\sin\theta \\ \frac{d^2V}{d\theta^2} &= \frac{I_3^2\omega_3^2}{I}\frac{2+\cos\theta-\cos^2\theta}{\left(1+\cos\theta\right)^3} - Mgh\cos\theta \end{aligned}$$

At $\theta = 0$, $\frac{dV}{d\theta} = 0$, as expected since $\theta = 0$ is an equilibrium point. In order for it to be stable, we need $\frac{d^2V}{d\theta^2} > 0$ at $\theta = 0$:

$$\begin{aligned} \frac{d^2V}{d\theta^2}_{\theta=0} &= \frac{I_3^2\omega_3^2}{4I} - Mgh > 0 \\ & \frac{4IMgh}{I_3^2\omega_3^2} < 1 \; . \end{aligned}$$